Set $S = \mathcal{O}_K \setminus \mathfrak{p}$

$$\Rightarrow \quad S^{-1}(\mathcal{O}_K[\alpha]) \simeq S^{-1}\left(\tfrac{1}{d}\mathcal{O}_K[\alpha]\right) \quad (\supseteq S^{-1}(\mathcal{O}_L) \text{ above})$$

$$\& \quad k(\mathfrak{p})[x]/\bar{f}(x) \simeq S^{-1}\left(k(\mathfrak{p})[x]/\bar{f}(x)\right)$$

$$\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \simeq S^{-1}\left(\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L\right)$$

$$\Rightarrow \quad \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \simeq \mathcal{O}_K[\alpha]/\mathfrak{p}$$

$$\begin{array}{cc} \wr\!\wr & \wr\!\wr \\ S^{-1}(\ldots) & S^{-1}(\ldots) \end{array} \qquad \square$$

Thm: $0 \neq \mathfrak{p} \subseteq \mathcal{O}_K$ prime

TFAE: i) $\mathfrak{p}$ unramified

ii) $\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$ is reduced, i.e. contains no nilpotent elts

iii) The $k(\mathfrak{p})$-bilin. pairing

$\mathrm{Tr}: \mathcal{O}_L/\mathfrak{p} \times \mathcal{O}_L/\mathfrak{p} \to k(\mathfrak{p})$ is non-deg.

If $K = \mathbb{Q}$,

iv) $p \nmid \Delta_L$

Proof: i) $\Leftrightarrow$ ii) : $\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \overset{\text{CRT}}{\simeq} \prod_{i=1}^{g} \mathcal{O}_L/\mathfrak{q}_i^{e_i}$

ii) $\Leftrightarrow$ iii) $\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \simeq \prod_{i=1}^{g} k(\mathfrak{q}_i)$

$k(\mathfrak{q}_i)/k(\mathfrak{p})$ sep. (bec. $k(\mathfrak{p})$ finite)

$\Rightarrow$ apply ex.

iii) $\Leftrightarrow$ iv) follows by definition

$\Delta_{L} = \det(\mathrm{Tr}_{L/\mathbb{Q}}(\alpha^i \alpha^j))$

Prop: $M/L/K$ fin. ext.

$\mathfrak{q}_M \subseteq \mathcal{O}_M$ maximal,

$\mathfrak{q}_L = \mathfrak{q}_M \cap \mathcal{O}_L$

$\mathfrak{p} = \mathfrak{q}_M \cap \mathcal{O}_K = \mathfrak{q}_L \cap \mathcal{O}_K$

$\Rightarrow$ i) $f(\mathfrak{q}_M|\mathfrak{p}) = f(\mathfrak{q}_M|\mathfrak{q}_L) \cdot f(\mathfrak{q}_L|\mathfrak{p})$

ii) $e(\mathfrak{q}_M|\mathfrak{p}) = e(\mathfrak{q}_M|\mathfrak{q}_L) \cdot e(\mathfrak{q}_L|\mathfrak{p})$

Prf: Exercise

(for i) $k(\mathfrak{p}) \hookrightarrow k(\mathfrak{q}_L) \hookrightarrow k(\mathfrak{q}_M)$)

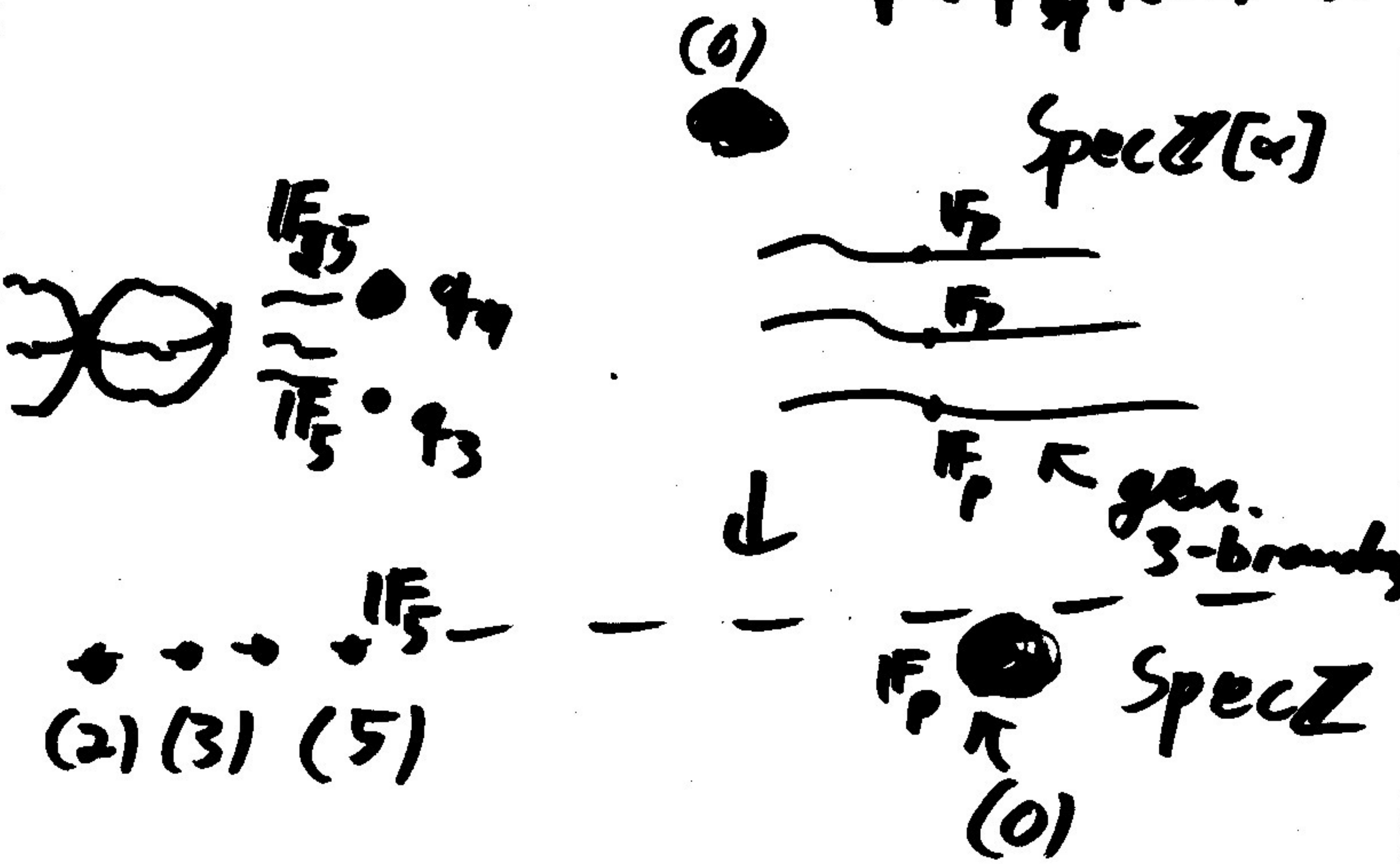
Geometric intuition:

$\mathbb{Z}[\alpha]$, $\alpha^3=2$

$(2)=\mathfrak{q}_1^3$, $(3)=\mathfrak{q}_2^3$,

$(5)=\mathfrak{q}_3\cdot\mathfrak{q}_4$, $f(\mathfrak{q}_3|(5))=1$

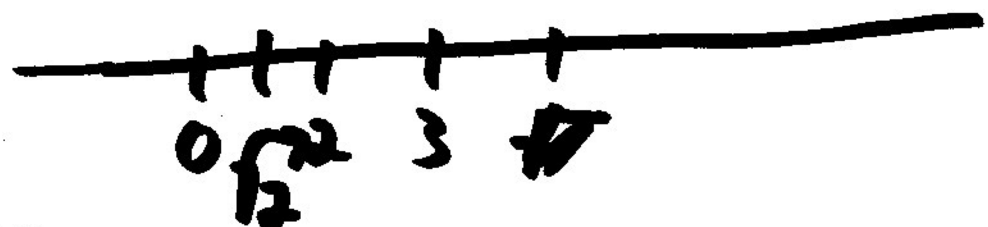
$f(\mathfrak{q}_4|(5))=2$



Spec $k[x,y]$

$y^2 - x^3 + x$ = B

↓



Spec $k[x]$

k = [redacted] A

$$\Theta/(\delta_{B/A}) \simeq \Omega^1_{B/A} = \{ b \cdot dc \mid b, c \in B \}/\sim$$

$$d(c_1 \cdot c_2) = c_1 \cdot dc_2 + c_2 \cdot dc_1$$

$d(a) = 0$, $a \in A$